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ASYMPTOTIC STABILITY OF AN OVERHEAD CRANE WITH NONLINEAR CONTROL AND VISCOUS DAMPING

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Abstract: Overhead cranes play a crucial role in various industrial applications, including construction, heavy component transport, and machinery handling. In this paper, we study the asymptotic stability of an overhead crane model governed by a semilinear evolution equation. The model incorporates a nonlinear feedback control and a viscous damping term, both of which significantly impact the stability properties. Our analysis focuses on the mathematical formulation, well-posedness, and stability behavior of the system.

Keywords and Phrases: Overhead crane, Nonlinear feedback, Semilinear evolution equations, Asymptotic stability.

2020 Mathematics Subject Classification: 35B35, 35B45, 35L20, 47D06, 93C20, 74S05.

1. Introduction

In this work, we investigate the asymptotic stability of an overhead crane system consisting of a trolley of mass m, controlled in position and velocity at $x = 0$. The trolley moves along a rail under the influence of a nonlinear force F, while a flexible cable of length l, with variable tension D, supports a heavy load of mass M. This setup can be found in various applications, such as overhead wire current-taking devices in trolleybuses, streetcars, and railways, where an orientable rod fitted with a grooved roller or sliding contact transmits electrical power. A similar configuration exists in moving carriages, where a suspended mass behaves like a pendulum (see Belunce et al. [1] and references therein). In such cases, the moving motor is often assumed to be frictionless.

Several studies have analyzed the linear behavior of such structures, particularly focusing on strong and uniform stability using Lyapunov's method, Shkalikov's approach, and Huang's method (see, e.g., [3, 5, 10, 11, 12]). For instance, Conrad and Mifdal [3] demonstrated that a hybrid system modeling an overhead crane with linear boundary conditions is strongly stable but not uniformly stable when the control depends solely on the trolley's position $u(0, t)$ and velocity $u_t(0, t)$. To achieve uniform stability, the cable's rotational velocity at $x = 0$ must be incorporated into the control law (see Mifdal [7]).

However, when subjected to large displacements due to severe loading, linear models fail to capture the true dynamic behavior of the system. In such cases, the assumption of small perturbations is no longer valid, and geometric nonlinearities must be considered for accurate modeling. These nonlinearities significantly alter the system's behavior compared to its linear counterpart.

Despite the importance of nonlinear boundary control in engineering applications, such studies remain rare in the literature. In the nonlinear setting, Saouri [10] examined the strong stability of an overhead crane model under velocity-dependent nonlinear feedback control.

This paper introduces two major novelties compared to previous works:

- A nonlinear feedback control law, incorporating both velocity and position dependence:
$$F(t) = -f(u_t(t, 0)) - g(u(t, 0)).$$
- A distributed viscous damping term $\xi(x)u_t(t, x)$, which affects the system's total energy and plays a crucial role in stability analysis.

Unlike previous studies, we adopt a different methodological approach to extend Saouri's results [10] and further explore the stability properties of the crane system under nonlinear feedback.

The principle of virtual work states that the total work of internal and external forces is always equal to the work of acceleration forces for any infinitesimal virtual displacement δu (see Holzapfel [4]). In this model, the virtual work of external forces is given by:

$$\forall t > 0, \quad \forall \delta u, \quad \delta \mathcal{W}_e = -f(u_t(t,0))\delta u(t,0) - g(u(t,0))\delta u(t,0).$$

The governing partial differential equations (PDEs) for the trolley-cable-load system are:

$$\begin{aligned} u''(t,x) - (D(x)u_x(t,x))_x + \xi(x)u'(t,x) &= 0, \quad t > 0, \quad 0 < x < 1, &(1.1)\\ -D(0)u_x(t,0) + mu''(t,0) &= F(t), \quad t > 0, &(1.2)\\ D(1)u_x(t,1) + Mu''(t,1) &= 0, \quad t > 0, &(1.3)\\ u(0,x) = u_0(x), \quad u'(0,x) &= u_1(x), \quad 0 < x < 1, &(1.4)\end{aligned}$$

where $' = \frac{d}{dt}$ denotes the time derivative, and the subscript x represents the spatial derivative. Moreover,

- $u(t,x)$ represents the cable's displacement along the curvilinear abscissa x.
- $\xi(x) > 0$ is a positive damping function, ensuring energy dissipation.
- $D(x)$ is the variable tension in the cable, belonging to the Sobolev space $H^1(0,1)$ with $D(x) > D(0) > 0$.
- $F(t)$ is the nonlinear feedback force applied at $x = 0$.

The functions f and g satisfy the following properties:

- f is a monotone increasing function, belongs to $C^2(\mathbb{R})$, and satisfies $f(0) = 0$.
- $yf(y) > 0$ for $y \neq 0$, ensuring energy dissipation.
- The function $f(y)$ satisfies the following growth condition:

$$|f(y)| \geq Cy^2, \quad \text{for } |y| \leq \zeta, \tag{1.5}$$

  where $C, \zeta > 0$.
- $g \in C^2(\mathbb{R})$ and satisfies:

$$\int_0^{\psi} g(z)dz \geq 0, \quad \forall \psi \in \mathbb{R}. \tag{1.6}$$

The remainder of this paper is structured as follows. Section 2 establishes well-posedness of the system using the C_0-semigroup approach and analyzes Lyapunov stability. Section 3 proves trajectory precompactness for initial conditions in a dense subset of the Hilbert space. Section 4 characterizes the ω-limit sets and demonstrates that any classical solution tends to zero, ensuring asymptotic stability. The conclusions of our study are summarized in Section 5.

2. Preliminaries

We introduce the following Hilbert space:

$$\mathcal{H} = H^1(0,1) \times L^2(0,1) \times \mathbb{R}^2,$$

equipped with the inner product:

$$\langle z_1, z_2 \rangle_{\mathcal{H}} = \int_0^1 v_1 v_2 \, dx + \int_0^1 D(u_1)_x (u_2)_x \, dx + M\eta_1\eta_2 + m\chi_1\chi_2,$$

where $z_i = (u_i, v_i, \eta_i, \chi_i)^T$ for $i \in \{1,2\}$. We denote the associated norm by $\| \cdot \|_{\mathcal{H}}$. Next, we introduce the linear operator $\mathcal{A}_0 : D(\mathcal{A}_0) \subset \mathcal{H} \to \mathcal{H}$, with domain:

$$D(\mathcal{A}_0) = \left\{ z = (u, v, \eta, \chi)^T \in H^2(0,1) \times H^1(0,1) \times \mathbb{R}^2 \ : \ \eta = v(1), \quad \chi = v(0) \right\}. \tag{2.1}$$

The operator $\mathcal{A}_0$ is defined as follows:

$$\mathcal{A}_0 \begin{pmatrix} u \\ v \\ \eta \\ \chi \end{pmatrix} = \begin{pmatrix} v \\ \left(D(x)u_x(x)\right)_x - \xi(x)v \\ -\dfrac{D(1)}{M}u_x(1) \\ \dfrac{D(0)}{m}u_x(0) \end{pmatrix}. \tag{2.2}$$

We also introduce the nonlinear operator $\mathcal{A}_{fg}$ on $\mathcal{H}$, defined by:

$$\mathcal{A}_{fg} \begin{pmatrix} u \\ v \\ \eta \\ \chi \end{pmatrix} = \frac{1}{m} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -f(\chi) - g(u(0)) \end{pmatrix}. \tag{2.3}$$

Thus, the full operator governing the system, denoted by $\mathbb{A}$, is given by:

$$\mathbb{A} = \mathcal{A}_0 + \mathcal{A}_{fg}. \tag{2.4}$$

Notice that $\mathbb{A}$ is a nonlinear operator with domain $D(\mathbb{A}) = D(\mathcal{A}_0)$. Now, the system (1.1)-(1.4) can be rewritten as the following semilinear evolution problem in $\mathcal{H}$:

$$\begin{cases} \frac{d}{dt} z(t) = \mathbb{A} z(t), \quad t > 0, \\ z(0) = z_0 \in \mathcal{H}, \end{cases} \tag{2.5}$$

where

$$z(t) = (u(.,t), v(.,t), \eta(t), \chi(t))^T, \quad z(0) = (u_0, u_1, \eta_0, \chi_0)^T. \tag{2.6}$$

We now establish the well-posedness of the system by proving that $\mathcal{A}_0$ generates a C_0-semigroup of contractions.

Proposition 2.1. *The operator $\mathcal{A}_0$ defined by (2.1)-(2.2) generates a C_0-semigroup of contractions on $\mathcal{H}$, denoted by $\{T(t)\}_{t\geq 0}$.*

Proof. We first show that $\mathcal{A}_0$ is a dissipative operator. Let $z = (u, v, \eta, \chi)^T \in D(\mathcal{A}_0)$. We compute:

$$\begin{aligned} \langle \mathcal{A}_0 z, z \rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} v \\ (D(x)u_x(x))_x - \xi(x)v \\ -\frac{D(1)}{M} u_x(1) \\ \frac{D(0)}{m} u_x(0) \end{pmatrix}, \begin{pmatrix} u \\ v \\ \eta \\ \chi \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &= -\int_0^1 \xi(x) v^2 \, dx \leq 0, \end{aligned}$$

after integration by parts. Thus, $\mathcal{A}_0$ is dissipative. Next, we prove that $\mathcal{A}_0$ is maximal dissipative. Let $(\alpha, \mu, \nu, \sigma)^T \in \mathcal{H}$. We need to find a unique $z = (u, v, \eta, \chi)^T \in D(\mathcal{A}_0)$ such that:

$$(I - \mathcal{A}_0)(u, v, \eta, \chi)^T = (\alpha, \mu, \nu, \sigma)^T. \tag{2.7}$$

This is equivalent to solving the following system:

$$v = u - \alpha, \tag{2.8}$$

$$\mu = v - (D(x)u_x)_x + \xi(x)v, \tag{2.9}$$

$$\nu = \eta + \frac{D(1)}{M} u_x(1), \tag{2.10}$$

$$\sigma = \chi - \frac{D(0)}{m} u_x(0). \tag{2.11}$$

We define the function:

$$\Delta(x) = 1 + \xi(x), \quad x \in (0,1).$$

Multiplying (2.9) by a test function $\phi \in H^1(0,1)$ and integrating by parts, we obtain the weak formulation:

$$a(u,\phi) = l(\phi), \quad \forall \phi \in H^1(0,1), \tag{2.12}$$

where

$$a(u,\phi) = -D(1)u_x(1)\phi(1) + D(0)u_x(0)\phi(0) + \int_0^1 \Delta(x)u(x)\phi(x) + D(x)u_x(x)\phi_x(x)dx,$$
$$l(\phi) = \int_0^1 (\mu + \alpha\Delta(x))\phi dx.$$

Thus, by the Lax-Milgram theorem, there exists a unique $u \in H^1(0,1)$ such that (2.12) holds for all $\phi \in H^1(0,1)$. Since u is uniquely determined, v follows from (2.8), and η, χ from (2.10) and (2.11), ensuring that $I - \mathcal{A}_0$ is surjective. Thus, $\mathcal{A}_0$ is maximal dissipative. By the Lumer-Phillips theorem, $\mathcal{A}_0$ generates a C_0-semigroup of contractions on $\mathcal{H}$.

Due to Proposition 2.1 and the fact that $\mathcal{A}_{fg}$ is locally Lipschitz continuous (since it is continuously differentiable on $\mathcal{H}$), Theorem 1.2 in Pazy [9, p. 184] ensures the following result:

Proposition 2.2. *For any initial condition $z_0 \in \mathcal{H}$, there exists a unique mild solution $z \in C([0,T], \mathcal{D}(\mathbb{A})) \cap C^1([0,T), \mathcal{H})$ of (2.5) (where $T > 0$ depends on z_0), given by the variation of constants formula*:

$$z(t) = e^{t\mathcal{A}_0} z_0 + \int_0^t e^{(t-s)\mathcal{A}_0} \mathcal{A}_{fg}\big(z(s)\big)\, ds, \quad t \in (0,T). \tag{2.13}$$

Now, the total energy of the system (1.1)-(1.4) at time t is defined as:

$$E(t,u) = E_{\text{crane}}(t,u) + E_{\text{control}}(t,u), \tag{2.14}$$

where

$$E_{\text{crane}}(t,u) = \frac{1}{2}\left[\int_0^1 (u')^2 dx + \int_0^1 D(x)u_x^2 dx + M(u'(1))^2 + m(u'(0))^2\right], \tag{2.15}$$

and

$$E_{\text{control}}(t,u) = \int_0^{u(0)} g(\zeta) d\zeta. \tag{2.16}$$

The term E_{crane} represents the mechanical energy of the system without external damping. It is composed of the kinetic energy:

$$E_c = \frac{1}{2}\left[\int_0^1 (u')^2 dx + M(u'(1))^2 + m(u'(0))^2\right],$$

and the elastic potential energy:

$$E_p = \frac{1}{2}\int_0^1 D(x)u_x^2 dx.$$

Consequently, the dissipation of the total energy is given by:

$$\frac{d}{dt}E(t,u) = -f(u'(0))u'(0) - \int_0^1 \xi(x)(u'(t,x))^2\, dx \le 0. \tag{2.17}$$

Equation (2.17) implies that the energy is decreasing, making it a good candidate for a Lyapunov functional for (2.5). We consider the following Lyapunov function:

$$\mathcal{V}(z(t)) = \frac{1}{2}\int_0^1 (u')^2 dx + \frac{1}{2}\int_0^1 D(x)u_x^2 dx + \int_0^{u(0)} g(\zeta)d\zeta + \frac{M}{2}(u'(1))^2 + \frac{m}{2}(u'(0))^2. \tag{2.18}$$

The functional $\mathcal{V}$ is a Lyapunov function for any initial condition $z_0 \in \mathcal{H}$. Indeed, since z is a classical solution on $[0,T]$, we can differentiate $\mathcal{V}$ along the classical solutions and obtain:

$$\frac{d}{dt}\mathcal{V}(z(t)) = -f(u'(0))u'(0) - \int_0^1 \xi(x)(u'(t,x))^2\, dx \le 0. \tag{2.19}$$

Thus, for any initial condition $z_0 \in \mathcal{H}$, (2.5) has a unique global mild solution. The family of operators $\{S(t)\}_{t\ge 0}$ defined on $\mathcal{H}$ by:

$$S(t)z_0 = z(t) \tag{2.20}$$

is a strongly continuous semigroup of nonlinear operators in $\mathcal{H}$ (see Pazy [9]). Hence, the system (2.5) is Lyapunov stable.

3. Precompactness of the trajectories

Lemma 3.1. *Suppose that $z(t) \in C^2([0,\infty[,\mathcal{H})$, then it follows that $\Psi(t) = \frac{d}{dt}z(t) \in C^1([0,\infty[,\mathcal{H})$. The evolution problem*

$$\frac{d}{dt}\Psi(t) = A_0\Psi(t) + \tilde{\mathcal{A}}_{fg}(t,\Psi), \tag{3.1}$$

$$\Psi(0) = \Psi_0 \in \mathcal{H}, \tag{3.2}$$

corresponds to the following PDE system:

$$u'''(t,x) - (D(x)u_x)'_x(t,x) + \xi(x)u''(t,x) = 0, \quad x \in (0,1), \quad t \geq 0, \tag{3.3}$$

$$-D(0)u'_x(t,0) + mu'''(t,0) = -u''(t,0)f'(u'(t,0)) - u'(t,0)g'(u(t,0)), \quad t \geq 0 \tag{3.4}$$

$$D(1)u'_x(t,1) + Mu'''(t,1) = 0, \quad t \geq 0 \tag{3.5}$$

$$u'(0,x) = u'_0(x), \quad u''(0,x) = u'_1(x), \quad x \in (0,1). \tag{3.6}$$

admits a mild solution $\Psi = [v, u'', \eta', \chi']^T \in \mathcal{H}$ *for any initial condition* $\Psi(0) = \Psi_0 \in \mathcal{H}$. *Furthermore, if* $\Psi_0 \in \mathcal{D}(\mathbb{A})$ *(hence,* $z_0 \in \mathcal{D}(\mathbb{A}^2)$*), then* $\Psi(t)$ *is a classical solution.*

Proof. We begin by differentiating system (2.5) with respect to time. The function Ψ satisfies the system (3.1)-(3.2), where:

$$\tilde{\mathcal{A}}_{fg}(t,\Psi) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\chi' f'(\chi) - \chi g'(u(0)) \end{bmatrix}. \tag{3.7}$$

Each term in $\tilde{\mathcal{A}}_{fg}(t,\Psi)$ involves functions of the form $\chi g'(u(0))$ and $\chi' f'(\chi)$. From Proposition 2.2, we know that $z(t) \in C^1([0,T),\mathcal{H})$, meaning that $\chi(t)$ and $u(t,0)$ are continuously differentiable in time. Moreover, since $\Psi(t) \in C^1([0,T),\mathcal{H})$, it follows that $\chi'(t)$ is also continuously differentiable in time. Since f' and g' are assumed to be C^1, their compositions with these functions remain continuously differentiable. Therefore, by the continuity of differentiation under compositions and products, we conclude that:

$$\tilde{\mathcal{A}}_{fg} \in C^1([0,+\infty[,\mathcal{H}).$$

Thus, $\tilde{\mathcal{A}}_{fg}$ is continuously differentiable in time. Furthermore, it is easy to prove that $\tilde{\mathcal{A}}_{fg}$ is Lipschitz continuous in $\mathcal{H}$, uniformly for $t \in [0,T]$ for any $T > 0$. By applying Theorem 1.2 in Pazy [9], p. 184, we conclude that there exists a unique global mild solution $\Psi(t)$ of (3.1)-(3.2) for every $\Psi_0 \in \mathcal{H}$, given by the variation of constants formula:

$$\Psi(t) = e^{tA_0}\mathbb{A}(z_0) + \int_0^t e^{(t-\epsilon)A_0}\frac{d}{d\epsilon}\mathcal{A}_{fg}(z(\epsilon))\,d\epsilon.$$

This result follows from the differentiation of equation (2.13) and the application of Corollary 2.5 in Pazy [9], p. 107.

Finally, $\tilde{\mathcal{A}}_{fg}$ is continuously differentiable and if $z_0 \in \mathcal{D}(\mathbb{A}^2)$, then by Theorem 1.5 in Pazy [9], p. 187, $\Psi(t)$ is a classical solution, and thus it satisfies the problem (3.3)-(3.6).

The goal of the following theorem is to establish the precompactness of the trajectory $\gamma(z_0)$ in $\mathcal{H}$. Specifically, we show that for any initial condition $z_0 \in D(\mathbb{A}^2)$, the trajectory remains in a bounded subset of $D(\mathbb{A})$, namely

$$\mathcal{B} = \{\Psi \in D(\mathbb{A}) \mid \|\Psi\|_{\mathcal{H}} \leq \epsilon\},$$

where $\epsilon > 0$ is a constant independent of t. Since $D(\mathbb{A})$ is compactly embedded in $\mathcal{H}$, this implies that every sequence along the trajectory admits a convergent subsequence, proving its precompactness.

Theorem 3.2. *For $z_0 \in D(\mathbb{A}^2)$, the trajectory $\gamma(z_0) := \cup_{t\geq 0} S(t) z_0$ is precompact.*

Proof. To prove the precompactness of the trajectory $\gamma(z_0)$, it suffices to show that

$$\mathcal{B} = \{\Psi \in D(\mathbb{A}) \mid \|\Psi\|_{\mathcal{H}} \leq \epsilon(\|z_0\|_{\mathcal{H}}, \|\Psi_0\|_{\mathcal{H}})\}, \quad \epsilon > 0,$$

is uniformly bounded over time. This is equivalent to proving that the energy-like functional $\mathcal{V}(\Psi(t))$ remains uniformly bounded over time (see Lemma 3.3 in [8]). By multiplying equation (3.3) by u'', integrating by parts over $[0,1]$, and using the boundary conditions, we obtain:

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(\Psi(t)) &= u''(t,0)g(u'(t,0)) - u''(t,0)g'(u(t,0))u'(t,0) \\ &\quad - f'(u'(t,0))(u''(t,0))^2 - \int_0^1 \xi(x)(u''(t,x))^2\,dx, \quad \forall t \geq 0. \end{aligned} \tag{3.8}$$

Since the last two terms are non-positive due to the assumptions on f and ξ, it follows that:

$$\frac{d}{dt}\mathcal{V}(\Psi(t)) \leq u''(t,0)g(u'(t,0)) - u''(t,0)g'(u(t,0))u'(t,0), \quad \forall t \geq 0. \tag{3.9}$$

Integrating over $[0,t]$, we get:

$$\mathcal{V}(\Psi(t)) \leq \mathcal{V}(\Psi(0)) + \int_0^t u''(\tau,0)g(u'(\tau,0))d\tau - \int_0^t u''(\tau,0)g'(u(\tau,0))u'(\tau,0)d\tau. \tag{3.10}$$

Rewriting the first integral:

$$\int_0^t u''(\tau,0)g(u'(\tau,0))d\tau = \int_{u'(0,0)}^{u'(t,0)} g(w)dw.$$

To ensure this integral is uniformly bounded, we justify that $u'(t,0)$ is uniformly bounded. This follows from the Lyapunov function:

$$\mathcal{V}(z(t)) = \frac{1}{2}\left[\int_0^1 (u')^2 dx + \int_0^1 D(x)u_x^2 dx + M(u'(1))^2 + m(u'(0))^2\right] + \int_0^{u(0)} g(\zeta)d\zeta.$$

Since $\frac{d}{dt}\mathcal{V}(z(t)) \leq 0$, we obtain $\mathcal{V}(z(t)) \leq \mathcal{V}(z_0)$ for all $t \geq 0$, ensuring that $u'(t,0)$ remains uniformly bounded. Since g is continuous, the integral remains bounded. For the second integral, using $u''(\tau,0)u'(\tau,0) = \frac{1}{2}\frac{d}{d\tau}(u'(\tau,0))^2$, integration by parts gives:

$$\int_0^t u''(\tau,0)\, g'(u(\tau,0))\, u'(\tau,0)\, d\tau = \left[\tfrac{1}{2}\,(u'(\tau,0))^2\, g'(u(\tau,0))\right]_0^t - \int_0^t \tfrac{1}{2}\,(u'(\tau,0))^3\, g''(u(\tau,0))\, d\tau.$$

Since $u'(\tau,0)$ and $g'(u(\tau,0))$ are bounded, the first term remains bounded. For the second term, the Sobolev embedding $H^2(0,1) \hookrightarrow C(0,1)$ gives:

$$\left|\int_0^t \frac{1}{2}(u'(\tau,0))^3 g''(u(\tau,0))d\tau\right| \leq C\int_0^t |u'(\tau,0)|^3 d\tau.$$

Using $|f(u'(t,0))| \geq Cu'(t,0)^2$, we deduce:

$$\int_0^{+\infty} |u'(t,0)|^3 dt \leq \frac{1}{C}\int_0^{+\infty} f(u'(t,0))u'(t,0)dt.$$

Since $\dfrac{d}{dt}\mathcal{V}(\Psi(t))$ is integrable on $[0,+\infty[$, it follows that $u'(t,0) \in L^3(0,+\infty)$, meaning that the second integral remains uniformly bounded. Thus, we conclude that $\mathcal{V}(\Psi(t))$ is uniformly bounded, and therefore, by Lemma 3.3 in [8], the function $t \mapsto \| \Psi(t) \|_{\mathcal{H}}$ is also uniformly bounded. Now, since $u(t,\cdot)$ is bounded in $H^2(0,1)$, and using the compact embeddings

$$H^2(0,1) \hookrightarrow\hookrightarrow H^1(0,1) \hookrightarrow\hookrightarrow L^2(0,1),$$

we deduce that the set of trajectories $\gamma(z_0)$ is relatively compact in $\mathcal{H}$. Since $\mathcal{V}(z(t)) \leq \mathcal{V}(z_0)$ for all $t \geq 0$, it follows that the bound ϵ depends only on the initial norms $\|z_0\|_{\mathcal{H}}$ and $\|\Psi_0\|_{\mathcal{H}}$, but not on time. By the compact embedding property, every bounded sequence in $D(\mathbb{A})$ admits a convergent subsequence in $\mathcal{H}$. Thus, $\gamma(z_0)$ is precompact.

We have previously established the precompactness of the trajectory $\gamma(z_0)$ when $z_0 \in D(\mathbb{A}^2)$. We now aim to generalize this result to the broader case where $z_0 \in D(\mathbb{A})$, proving that the associated trajectory remains in a relatively compact subset of $\mathcal{H}$. This generalization relies on an approximation by elements in $D(\mathbb{A}^2)$ and the use of compact embeddings adapted to this level of regularity. We begin by establishing the density of $D(\mathbb{A}^2)$ in $D(\mathbb{A})$, which is essential for extending the results previously obtained for $D(\mathbb{A}^2)$ to the more general case of $D(\mathbb{A})$.

Lemma 3.3. *The space $D(\mathbb{A}^2)$ is dense in $D(\mathbb{A})$, i.e., for every $z \in D(\mathbb{A})$, there exists a sequence $(z_n) \subset D(\mathbb{A}^2)$ such that:*

$$z_n \to z \quad \text{in } D(\mathbb{A}), \tag{3.11}$$

$$\Psi_n = \mathbb{A}z_n \to \Psi = \mathbb{A}z \quad \text{in } \mathcal{H}. \tag{3.12}$$

Proof. Since $D(\mathbb{A})$ is given by:

$$D(\mathbb{A}) = \left\{ z = (u, v, \eta, \chi)^T \in H^2(0,1) \times H^1(0,1) \times \mathbb{R}^2 \ : \ \eta = v(1), \quad \chi = v(0) \right\},$$

we construct an approximating sequence $(z_n) \subset D(\mathbb{A}^2)$.

Since $H^3(0,1)$ is dense in $H^2(0,1)$, we can choose a sequence $(u_n) \subset H^3(0,1)$ such that:

$$u_n \to u \quad \text{in } H^2(0,1).$$

We define $v_n = u_n'$ and since $H^2(0,1)$ is dense in $H^1(0,1)$ ensuring that:

$$v_n \to v = u' \quad \text{in } H^1(0,1).$$

Since $v_n \in H^1(0,1)$, it follows that:

$$\eta_n = v_n(1) \to \eta \quad \text{in } \mathbb{R}, \quad \chi_n = v_n(0) \to \chi \quad \text{in } \mathbb{R}.$$

Thus, the sequence $z_n = (u_n, v_n, \eta_n, \chi_n)^T$ satisfies $z_n \to z$ in $D(\mathbb{A})$.

Now, we check the convergence of $\mathbb{A}z_n$. By definition,

$$\mathbb{A}z_n = \begin{pmatrix} v_n \\ (D(x)u_{n,x})_x - \xi(x)v_n \\ -\frac{D(1)}{M}u_{n,x}(1) \\ \frac{D(0)}{m}u_{n,x}(0) \end{pmatrix}.$$

Since $u_n \to u$ in $H^2(0,1)$ and $v_n \to v$ in $H^1(0,1)$, we have:

$$(D(x)u_{n,x})_x - \xi(x)v_n \to (D(x)u_x)_x - \xi(x)v \quad \text{in } H^1(0,1).$$

Moreover,

$$u_{n,x}(1) \to u_x(1) \quad \text{in } \mathbb{R}, \quad u_{n,x}(0) \to u_x(0) \quad \text{in } \mathbb{R},$$

ensuring that:

$$\mathbb{A} z_n \to \mathbb{A} z \quad \text{in } \mathcal{H}.$$

Thus, the sequence (z_n) satisfies (3.11) and (3.12), proving that $D(\mathbb{A}^2)$ is dense in $D(\mathbb{A})$.

Theorem 3.4. *For any $z_0 \in D(\mathbb{A})$, the trajectory $\gamma(z_0) := \cup_{t \geq 0} S(t) z_0$ is precompact in $\mathcal{H}$.*

Proof. According to Lemma 3.3, there exists a sequence $(z_{n0})_{n\in\mathbb{N}} \subset D(\mathbb{A}^2)$ such that:

$$\lim_{n\to+\infty} z_{n0} = z_0 \text{ and } \lim_{n\to+\infty} \mathbb{A} z_{n0} = \mathbb{A} z_0.$$

Set $z_n(t) = S(t) z_{n0}$. Since $\Psi_n(0) = \mathbb{A} z_{n0}$, it follows that:

$$\lim_{n\to+\infty} \Psi_n(0) = \mathbb{A} z_0 \quad \text{in} \quad \mathcal{H}. \tag{3.13}$$

Since $z_{n0} \to z_0$ in $D(\mathbb{A})$ and $\mathbb{A} z_{n0} \to \mathbb{A} z_0$ in $\mathcal{H}$, it follows that the sequences $(z_{n0})_{n\in\mathbb{N}}$ and $(\Psi_n(0))_{n\in\mathbb{N}}$ are uniformly bounded in $\mathcal{H}$.

By applying the uniform bound:

$$\sup_{t\geq 0,\ n\in\mathbb{N}} \| \Psi_n(t) \|_{\mathcal{H}} \leq C,$$

it follows that Ψ_n is bounded in $L^\infty([0,+\infty[,\mathcal{H})$. By the Banach-Alaoglu theorem (see Cazenave [2]), there exists $\tilde{z} \in L^\infty([0,+\infty[,\mathcal{H})$ and a subsequence $(z_{nk})_{k\in\mathbb{N}}$ such that:

$$\Psi_{nk} \overset{*}{\rightharpoonup} \tilde{z} \quad \text{in} \quad L^\infty([0,+\infty[,\mathcal{H}).$$

For any $y \in \mathcal{H}$ and $t \geq 0$, it follows that:

$$\lim_{k\to+\infty} \int_0^t \langle \Psi_{nk}(\tau), y \rangle_{\mathcal{H}} d\tau = \int_0^t \langle \tilde{z}(\tau), y \rangle_{\mathcal{H}} d\tau.$$

Thus,

$$\lim_{k\to+\infty} \langle z_{nk}(t) - z_{nk}(0), y \rangle_{\mathcal{H}} = \left\langle \int_0^t \tilde{z}(\tau) d\tau, y \right\rangle_{\mathcal{H}}.$$

Since $z_n(t) \to z(t)$ in $\mathcal{H}$ for all $t \geq 0$ (by a consequence of Proposition 4.3.7 in [2]), we obtain:

$$\langle z(t) - z(0), y \rangle_{\mathcal{H}} = \left\langle \int_0^t \tilde{z}(\tau) d\tau, y \right\rangle_{\mathcal{H}}.$$

Since y is arbitrary, it follows that:

$$z(t) - z(0) = \int_0^t \tilde{z}(\tau) d\tau. \tag{3.14}$$

Differentiating (3.14), we obtain $\Psi \equiv \tilde{z}$, ensuring that $z \in C^1([0, +\infty[, \mathcal{H})$, so that $\Psi \in L^\infty([0, +\infty[, \mathcal{H})$, implying that $\| \Psi(t) \|_{\mathcal{H}}$ is uniformly bounded.

Since $z(t)$ is bounded in $D(\mathbb{A})$ and using the compact embedding:

$$D(\mathbb{A}) \hookrightarrow\hookrightarrow \mathcal{H},$$

we conclude that any sequence extracted from $\gamma(z_0)$ admits a convergent subsequence in $\mathcal{H}$, proving precompactness.

4. Characterization of the ω-limit set and asymptotic stability

The goal of this section is to characterize the set of possible accumulation points of the system's trajectories as time tends to infinity. We introduce the ω-limit set $\omega(z_0)$, which consists of all possible accumulation points of the trajectory $S(t)z_0$ as $t \to +\infty$. A key property in [2] of $\omega(z_0)$ is its invariance under the semigroup $S(t)$, ensuring that any trajectory starting from an element of $\omega(z_0)$ remains within the set. Additionally, the Lyapunov function $\mathcal{V}$ is shown to be non-increasing along the trajectories and converges to a limit $\nu(z_0)$ as $t \to +\infty$.

To identify ω-limit sets, we analyze trajectories along which $\mathcal{V}$ remains constant. We introduce the largest S-invariant subset of the set where $\frac{d}{dt}\mathcal{V}(t)$ vanishes, denoted by Ω, and establish the inclusion:

$$\omega(z_0) \subset \Omega, \quad \forall z_0 \in \mathcal{H}.$$

This inclusion provides a framework for further characterizing the asymptotic behavior of the system. The set Ω is defined as the largest S-invariant subset of the states where the Lyapunov function $\mathcal{V}$ remains constant:

$$\Omega = \left\{ z = (u, v, \eta, \chi) \in \mathcal{H} \;\middle|\; \frac{d}{dt}\mathcal{V}(z) = 0 \right\}.$$

Proposition 4.1. *Under the given assumptions on f and ξ, the set Ω is given by:*

$$\Omega = \left\{ (u_\infty, 0, 0, 0) \in \mathcal{H} \;\middle|\; u_\infty \in \mathbb{R} \right\}.$$

Moreover, if $u(0, t) = 0$ for all $t \geq 0$, then:

$$\Omega = \{0\}.$$

Proof. From the dissipation equation:

$$\frac{d}{dt}V(z(t)) = -f(u'(0))u'(0) - \int_0^1 \xi(x)(u'(x))^2 dx, \tag{4.1}$$

and the fact that Ω is S-invariant, we must have:

$$f(u'(0))u'(0) + \int_0^1 \xi(x)(u'(x))^2 dx = 0. \tag{4.2}$$

Step 1: Implication for $u'(0)$

Since $f(y)$ is monotone increasing and satisfies $yf(y) > 0$ for $y \neq 0$, it follows that:

$$f(u'(0))u'(0) > 0 \quad \text{if } u'(0) \neq 0.$$

This contradicts equation (4.2), implying that:

$$u'(0) = 0.$$

Step 2: Implication for $u'(x)$ in $(0, 1)$

Since $\xi(x) > 0$ for all $x \in (0, 1)$, if $u'(x) \neq 0$ on a non-negligible subset of $(0, 1)$, then:

$$\int_0^1 \xi(x)(u'(x))^2 dx > 0.$$

Again, this contradicts equation (4.2), so we conclude that:

$$u'(x) = 0, \quad \forall x \in (0, 1).$$

Step 3: Conclusion on u and on χ, η, v

Since $u'(x) = 0$, it follows that $u(x)$ is constant, i.e., $u(x) = u_\infty$ for some $u_\infty \in \mathbb{R}$.

From the boundary conditions:

$$\chi = u'(0) = 0, \quad \eta = u'(1) = 0.$$

Since $v = u'$, we also conclude that:

$$v = 0.$$

Step 4: Condition for $u_\infty = 0$

If we further assume $u(0) = 0$, then the only admissible constant solution is $u_\infty = 0$, leading to:

$$\Omega = \{0\}.$$

Thus, the only possible elements in Ω are:

$$\Omega = \left\{ (u_\infty, 0, 0, 0) \in \mathcal{H} \;\middle|\; u_\infty \in \mathbb{R} \right\}.$$

Theorem 4.2. *For all $z_0 \in D(\mathcal{A})$,*

$$\lim_{t \to +\infty} z(t) = 0.$$

That is, the system (2.5) is asymptotically stable.

Proof. From the previous section, the trajectory $\gamma(z_0)$ is precompact, implying that $\omega(z_0)$ is nonempty. Moreover, we established that:

$$\omega(z_0) \subset \Omega.$$

If $\Omega = \{0\}$, then necessarily:

$$\omega(z_0) = \{0\}.$$

This means that there exists a sequence $(t_n)_{n \in \mathbb{N}}$ such that:

$$\lim_{n \to +\infty} t_n = +\infty, \quad \text{and} \quad \lim_{n \to +\infty} z(t_n) = 0.$$

Since $\mathcal{V}$ is decreasing along trajectories, we have:

$$\lim_{n \to +\infty} \mathcal{V}(z(t_n)) = \mathcal{V}\left(\lim_{t \to +\infty} z(t_n) \right) = 0.$$

Since $\mathcal{V}$ is a continuous function and non-increasing along trajectories, its limit as $t \to +\infty$ must also be 0:

$$\lim_{t \to +\infty} \mathcal{V}(z(t)) = 0.$$

Since $\mathcal{V}(z)$ is positive definite, this implies:

$$\lim_{t \to +\infty} z(t) = 0.$$

5. Conclusion

In this work, we analyzed the asymptotic behavior of a controlled system governed by a partial differential equation with boundary damping. By constructing a suitable Lyapunov function, we established its monotonicity and derived key dissipation properties that allowed us to study the long-term evolution of the system. An important step in our analysis was the characterization of the ω-limit set, which consists of all possible accumulation points of the trajectories as time tends to infinity. By leveraging the invariance properties of this set and using the dissipation equation, we showed that under the given conditions on f and ξ, all solutions converge to a set of equilibrium states. More specifically, we demonstrated that the velocity and boundary terms vanish asymptotically, reducing the system's dynamics to a stationary state. Furthermore, we proved that if the additional condition $u(0,t) = 0$ for all $t \geq 0$ holds, then the only possible equilibrium is the trivial solution, ensuring strict asymptotic stability. Our results highlight the fundamental role of energy dissipation in stabilizing the system and provide a framework for ensuring that all trajectories converge to equilibrium.

References

[1] Belunce A., Pandolfo V., Roozbahani H. and Handroos H., Novel control method for overhead crane's load stability, Dyn. Vibroacoust. Mach., Procedia Eng., 106 (2015), 108-125.

[2] Cazenave T. and Haraux A., An Introduction to semilinear evolution equations, Oxford Lecture Ser. Math. Appl., The Clarendon Press, Oxford Univ. Press, New York, 13 (1998).

[3] Conrad F. and Mifdal A., Strong stability of a model of an overhead crane, Control Cybern., 27(3) (1998), 363-374.

[4] Holzapfel G. A., Nonlinear Solid Mechanics, J. Wiley & Sons, Chichester, England, 2000.

[5] Littman W. and Marcus L., Stabilization of a hybrid system of elasticity by feedback boundary damping, Ann. Mat. Pura Appl., 152 (1988), 281-330.

[6] Luo Z. H., Guo B. Z. and Morgül O., Stability and Stabilization of Infinite Dimensional Systems with Applications, Commun. Control Eng. Ser., Springer-Verlag, London, 1999.

[7] Mifdal A., Étude de la stabilisation forte et uniforme d'un système hybride: Application à un modèle de pont roulant, Thèse, Univ. Nancy, 1997.

[8] Miletic M., Stürzer D. and Arnold A., An Euler-Bernoulli beam with nonlinear damping and a non linear spring at the tip, arXiv:1411.7946 [math.AP], (2015).

[9] Pazy A., Semigroups of linear operators and applications to partial differential equations, Appl. Math. Sci., vol. 44, Springer-Verlag, New York, 1983.

[10] Saouri F., Stabilisation de quelques systèmes élastiques. Analyse spectrale et comportement asymptotique, Thèse de Doctorat, Univ. Henri Poincaré I, 2000.

[11] Stürzer D., Arnold A. and Kugi A., Closed-loop stability analysis of a gantry crane with heavy chain, arXiv:1607.07052v1 [math.AP], (2016).

[12] Touré K. A., Mensah E. P. and Taha M. M., Stabilization and Riesz basis property for an overhead crane model with feedback in velocity and rotating velocity, J. Nonlinear Anal. Appl., (2014), 1-14.